ON REDUCTIONS OF SOME KDV-TYPE SYSTEMS AND THEIR LINK TO THE QUARTIC HE'NON-HEILES HAMILTONIAN

Bilinear integrable systems - from classical to quantum, continuous to discrete, Kluwer. NATO ARW, 15–19 September 2002, Isola d'Elba. S2003/097.

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26 June 2003

Abstract. A few 2+1-dimensional equations belonging to the KP and modified KP hierarchies are shown to be sufficient to provide a unified picture of all the integrable cases of the cubic and quartic Hénon-Heiles Hamiltonians.

1. Introduction

The Hénon-Heiles (HH) Hamiltonian [14] with a generalized cubic potential is defined as

$$\mathrm{HH3}: H = \frac{1}{2}(p_1^2 + p_2^2 + c_1 q_1^2 + c_2 q_2^2) + \alpha q_1 q_2^2 - \frac{\beta}{3} q_1^3 + \frac{c_3}{q_2^2}, \qquad (1)$$

in which $\alpha, \beta, c_1, c_2, c_3$ are constants.

The corresponding equations of motion pass the Painlevé test for only three sets of values of the ratio β/α, which are also the only three cases for which an additional first integral K has been found [4, 5, 12]. These three cases have been integrated [7, 25] with genus-two hyperelliptic functions. Moreover, they are equivalent [11] to the stationary reduction of three fifth order soliton equations, called fifth order Korteweg de Vries (KdV_5), Sawada-Kotera (SK) and Kaup-Kupershmidt (KK) equations, belonging

respectively to the KP, BKP and CKP hierarchies whose Hirota bilinear forms can be found in [16].

If the potential is taken as the most general cubic polynomial in (q_1, q_2), there exists a fourth Liouville integrable case,

$$V = q_1^3 + \frac{1}{2} q_2^2 q_1 + \frac{i}{6\sqrt{3}} q_2^3, \tag{2}$$

detected by Ramani *et al.* [20], but up to now its general solution is unknown.

Another Hénon-Heiles-type Hamiltonian with an extended quartic potential has been considered,

$$\begin{aligned} \text{HH4} : H = & \frac{1}{2}(P_1^2 + P_2^2 + aQ_1^2 + bQ_2^2) + CQ_1^4 + BQ_1^2Q_2^2 + AQ_2^4 \\ & + \frac{1}{2}\left(\frac{\alpha}{Q_1^2} + \frac{\beta}{Q_2^2}\right) + \mu Q_1, \end{aligned} \tag{3}$$

in which $A, B, C, \alpha, \beta, \mu, a, b$ are constants. Again, the equations of motion pass the Painlevé test for only four values of the ratios $A : B : C$ [20, 13, 15], which happen to be the only known cases of Liouville integrability. However, it is not yet completely settled whether, in all four cases, the quartic Hamiltonian (3) displays the same pattern as the cubic Hamiltonian (1), i. e.

– the equations of motion can be integrated with hyperelliptic functions of genus two,
– there exists an equivalence with the stationary reduction of some partial differential equation (PDE) belonging to the KP, BKP and CKP hierarchies.

In this paper, we first summarize the results already established for the systems (1) and (3). We then establish new links between the coupled KdV (c-KdV) systems considered in [2] and some other ones [9, 16, 24] belonging to the BKP and CKP hierarchies. These links could be useful to find the explicit general solution without any restriction on the parameters other than those generated by the Painlevé test.

2. Already integrated cases

The four cases for which the quartic Hamiltonian passes the Painlevé test are,

(i) $A : B : C = 1 : 2 : 1, \mu = 0$. The system is then equivalent to the stationary reduction of the Manakov system [18] of two coupled nonlinear Schrödinger (NLS) equations and has been integrated [27] with genus two hyperelliptic functions.

(ii)$A:B:C=1:6:1, a=b, \mu=0$,
(iii) $A:B:C=1:6:8, a=4b, \alpha=0$,
(iv) $A:B:C=1:12:16, a=4b, \mu=0$.

Each of the last three cases is equivalent [2] to the stationary reduction of a coupled KdV system possessing a fourth or fifth order Lax pair. Canonical transformations have been found [2, 1] which allow us [26, 23] to define the separating variables of the Hamilton-Jacobi equation, however with additional restrictions on α, β, μ, as showed in Table 1.

TABLE 1. All the cases of HH3 and HH4 which pass the Painlevé test, with the extra terms c_3 or α, β, μ. First column indicates the cubic or quartic case. Second column is the value of β/α (if cubic) or the ratio $A:B:C$ (quartic), followed by the values selected by the Painlevé test. Third column indicates the polynomial degree of the additional constant of the motion K in the momenta (p_1, p_2). Next column displays the PDE system connected to the HH case. Last column shows the reference to the general solution and the not yet integrated cases. When the general solution is known, it is a singlevalued rational function of genus two hyperelliptic functions.

HH	case	deg K	PDE	General solution
3	$-1, c_1=c_2$	4	SK	[25]
3	-6, c_1, c_2 arb.	2	KdV_5	[7]
3	-16, $c_1=16c_2$	4	KK	[25]
4	$1:2:1$ $\mu=0$	2	c-NLS	[27]
4	$1:6:1$ $a=b, \mu=0$	4	c-KdV2, Lax order 4	$\alpha=\beta$[26], $\alpha\neq\beta$?
4	$1:6:8$ $a=4b, \alpha=0$	4	c-KdV1, Lax order 4	$\beta\mu=0$[26], $\beta\mu\neq 0$?
4	$1:12:16$ $a=4b, \mu=0$	4	c-KdVb Lax order 5	$\alpha\beta=0$[23], $\alpha\beta\neq 0$?

3. Link between KP hierarchies and integrable HH cases

Let us consider the following three systems of the KP and modified KP hierarchies [16],

$$\left\{\begin{array}{l} \left(D_1^4-4D_1D_3+3D_2^2\right)(\tau_0\cdot\tau_0)=0, \\ \left(\left(D_1^3+2D_3\right)D_2-3D_1D_4\right)(\tau_0\cdot\tau_0)=0, \end{array}\right. \tag{4}$$

$$\begin{cases} \left(D_1^4 - 4D_1D_3 + 3D_2^2\right)(\tau_0 \cdot \tau_0) = 0, \\ \left(D_1^6 - 20D_1^3D_3 - 80D_3^2 + 144D_1D_5 - 45D_1^2D_2^2\right)(\tau_0 \cdot \tau_0) = 0, \end{cases} \tag{5}$$

$$\begin{cases} \left(D_1^2 + D_2\right)(\tau_0 \cdot \tau_1) = 0, \\ \left(D_1^6 - 20D_1^3D_3 - 80D_3^2 + 144D_1D_5 \right. \\ \qquad \left. + 15\left(4D_1D_3 - D_1^4\right)D_2\right)(\tau_0 \cdot \tau_1) = 0, \end{cases} \tag{6}$$

in which the subscripts of the bilinear operators correspond to the components of the vector $\vec{x} = (x_1, x_2, \cdots, x_n)$, while τ_0 and τ_1 are functions of $\vec{x}$. By further putting some symmetry constraint on τ_0 and τ_1, let us define as follows four (2+1)-dimensional PDEs (see line "2+1" in Figure 1).

1. With the system (4), one defines by $D_4 = 0$ [21] the (2+1)-dim PDE labeled "KP-1" in Figure 1.
2. With the system (5), one defines by $D_2 = 0$ the (2+1)-dim PDE labeled "KP-2" in Figure 1.
3. With the system (6) and the B_∞ symmetry constraint [16, p. 968]

$$\begin{cases} \tau_0(x) = f(x_{\mathrm{odd}}) + x_2 g(x_{\mathrm{odd}}) + \frac{1}{2}x_2^2 h_1(x_{\mathrm{odd}}) + x_4 h_2(x_{\mathrm{odd}}) + \cdots, \\ \tau_1(x) = f(x_{\mathrm{odd}}) - x_2 g(x_{\mathrm{odd}}) + \frac{1}{2}x_2^2 h_1(x_{\mathrm{odd}}) - x_4 h_2(x_{\mathrm{odd}}) + \cdots, \end{cases} \tag{7}$$

   one defines the (2+1)-dim BKP equation

$$\begin{aligned} &9z_{x_1,x_5} - 5z_{2x_3} \\ &+\left(z_{5x_1} + 15z_{x_1}z_{3x_1} + 15\left(z_{x_1}\right)^3 - 5z_{2x_1,x_3} - 15z_{x_1}z_{x_3}\right)_{x_1} = 0, \end{aligned} \tag{8}$$

   in which $z = \partial_{x_1} \operatorname{Log} \tau_0(\vec{x})|_{x_2=x_4=\cdots=0}$ and $z_{2x_3} \equiv z_{x_3x_3} \ldots$.
4. With (5) and the C_∞ symmetry constraint [16, p. 968]

$$\tau_0(x) = f(x_{\mathrm{odd}}) + \frac{1}{2}x_2^2 g(x_{\mathrm{odd}}) + \frac{1}{2}x_4^2 h(x_{\mathrm{odd}}) + \cdots, \tag{9}$$

   one defines the (2+1)-dim CKP equation

$$\begin{aligned} &9z_{x_1,x_5} - 5z_{2x_3} \\ &+\left(z_{5x_1} + 15z_{x_1}z_{3x_1} + 15\left(z_{x_1}\right)^3 - 5z_{2x_1,x_3} - 15z_{x_1}z_{x_3} + \frac{45}{4}\left(z_{2x_1}\right)^2\right)_{x_1} = 0, \end{aligned} \tag{10}$$

   in which $z = \partial_{x_1} \operatorname{Log} \tau_0(\vec{x})|_{x_2=x_4=\cdots=0}$.

Next, from these (2+1)-dimensional PDEs, one performs the following natural reductions to (1+1)-dimensional PDEs (see line "1+1" in Figure 1).

1. In KP-1, the C_∞ symmetry constraint (9) defines
$$\left\{\begin{array}{l}\left(D_1^4-4D_1D_3\right)(f\cdot f)+6fg=0,\\ \left(D_1^3+2D_3\right)(f\cdot g)=0,\end{array}\right. \tag{11}$$
which we call bi-SH [21] for reasons explained in next section.
2. In KP-1, the constraint
$$\tau_0(x)=f(x_{\text{odd}})+x_2g(x_{\text{odd}})+\frac{1}{2}x_2^2h_1(x_{\text{odd}})+x_4h_2(x_{\text{odd}})+\cdots, \tag{12}$$
defines
$$\left\{\begin{array}{l}\left(D_1^4-4D_1D_3\right)(f\cdot f)-6g^2=0,\\ \left(D_1^3+2D_3\right)(f\cdot g)=0,\end{array}\right. \tag{13}$$
which is called coupled KdV system of Hirota-Satsuma (HSS) [21].
3. In KP-2, the elimination of x_3 [16, p. 962] yields the potential KdV_5 equation
$$z_t+z_{xxxxx}+5z_{xx}^2+10z_xz_{xxx}+10z_x^3=0, \tag{14}$$
with the notation $x\equiv x_1, t\equiv -x_5/16, z=2\partial_x\operatorname{Log}\tau_0$.
4. In BKP (8), the reduction $z_{x_3}=0$ defines the potential SK equation [22]
$$z_t+z_{xxxxx}+15z_xz_{xxx}+15\left(z_x\right)^3=0, \tag{15}$$
with the notation $x_5\equiv 9t, x_1\equiv x$.
5. In BKP (8), the reduction $z_{x_5}=0$ defines the 1+1-dimensional bi-SK or Ramani equation [19]
$$\left(z_{xxxxx}+15z_xz_{xxx}+15\left(z_x\right)^3-15z_xz_t-5z_{xxt}\right)_x-5z_{tt}=0, \tag{16}$$
with the notation $x_3\equiv t, x_1\equiv x$.
6. In CKP (10), the reduction $\partial_{x_3}\tau_0=0$ defines the fifth order potential KK equation [17]
$$z_t+z_{xxxxx}+15z_xz_{xxx}+15\left(z_x\right)^3+\frac{45}{4}\left(z_{xx}\right)^2=0, \tag{17}$$
with the notation $x_1\equiv x, x_5\equiv 9t$.
7. In CKP (10), the reduction $\partial_{x_5}\tau_0=0$ defines the sixth order bi-KK equation [10]
$$\begin{aligned}&\left(z_{xxxxx}+15z_xz_{xxx}+15\left(z_x\right)^3-15z_xz_t-5z_{xxt}+\frac{45}{4}\left(z_{xx}\right)^2\right)_x\\&-5z_{tt}=0,\end{aligned} \tag{18}$$
with the notation $x_1\equiv x, x_3\equiv t$.

Finally, the stationary reduction $(x,t) \to x - ct$ of these (1+1)-dimensional PDEs leads directly to the Hamiltonian systems or the ODE listed in the line "0+1" of Figure 1.

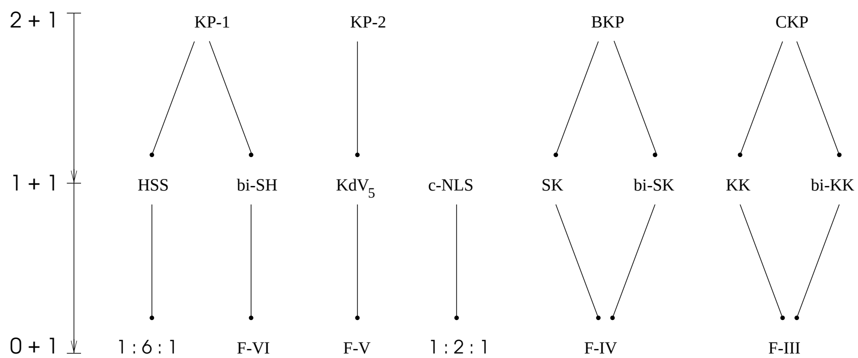

Figure 1. Reductions from (2+1)-dimensional PDEs to (1+1)-dimensional PDEs, then to ODEs (the notation F-xxx denotes the autonomous case of the ODE denoted F-xxx in [6]) or to Hamiltonian systems. The symbol c-NLS represents the Manakov system [18] of two coupled NLS equations.

The four ODEs F-III, F-IV, F-V, F-VI have a singlevalued general solution, obtained by the Jacobi postmultiplier method [6], which is expressed with genus-two hyperelliptic functions. Three of them (F-III, F-IV, F-V), which are the stationary reductions of respectively (17), (15), and (14), have been shown [11] to have a one-to-one correspondence with the q_1 component of the three integrable cases of HH3. Therefore the chain of reductions generated from the systems (5) and (6) contains the full information for the integration of HH3.

Let us now show that Figure 1 also contains the full information for the integration of HH4. This will involve two kinds of coupled KdV (c-KdV) systems: some with a fourth order Lax pair, some with a fifth order Lax pair.

4. Link of coupled KdV systems with HH4

In the variables $u = \partial_x^2 \operatorname{Log} f,\ v = 4g/f$, the bilinear system (11) is rewritten as the c-KdV system [8, 21, 16, 9]

$$\left\{ \begin{array}{l} -4u_t + \left(6u^2 + u_{xx} + 3v\right)_x = 0, \\ 2v_t + 6uv_x + v_{xxx} = 0, \end{array} \right. \tag{19}$$

with the notation $x_3 \equiv t, x_1 \equiv x$. This system possesses the fourth order Lax pair [21]

$$\left\{ \begin{array}{ll} \left(\partial_x^4 + 4u\partial_x^2 + 4u_x\partial_x + 2u_{xx} + 4u^2 + v\right)\psi & = \lambda\psi, \\ \left(\partial_x^3 + 3u\partial_x + \dfrac{3}{2}u_x\right)\psi & = \partial_t\psi, \end{array} \right. \tag{20}$$

Under the Miura transformation denoted M_3 in Figure 2

$$\left\{ \begin{array}{l} 4u = 2G - F_x - F^2, \\ 2v = 2F_{xxx} + 4FF_{xx} + 8GF_x + 4FG_x + 3F_x^2 - 2F^2F_x - F^4 + 4GF^2, \end{array} \right. \tag{21}$$

the system (19) is mapped to the following c-KdV system (denoted c-KdV$_1$ in Figure 2) given in [2, 1]

$$\left\{ \begin{array}{l} 4F_t = \left(-2F_{xx} - 3FF_x + F^3 - 6FG\right)_x, \\ 8G_t = 2G_{xxx} + 12GG_x + 6FG_{xx} + 12GF_{xx} + 18F_xG_x - 6F^2G_x \\ \qquad + 3F_{xxxx} + 3FF_{xxx} + 18F_xF_{xx} - 6F^2F_{xx} - 6FF_x^2, \end{array} \right. \tag{22}$$

with the Lax pair

$$\left\{ \begin{array}{l} \left(\partial_x^4 + (2G - F_x - F^2)\partial_x^2 + (2G - F_x - F^2)_x\partial_x \right. \\ \left. +(FG)_x + G_{xx} + G^2\right)\psi = \lambda\psi, \\ \left(\partial_x^3 + \dfrac{3}{4}(2G - F_x - F^2)\partial_x + \dfrac{3}{8}(2G - F_x - F^2)_x\right)\psi = \partial_t\psi, \end{array} \right. \tag{23}$$

The stationary reduction of this c-KdV$_1$ system happens to be the case 1:6:8 of HH4 for arbitrary values of (β, μ).

The field $z = \int u \mathrm{d}x$ of (19) satisfies the sixth order PDE [3],

$$-8z_{tt} + z_{xxxxxx} - 2z_{xxxt} + 18z_xz_{xxxx} + 36z_{xx}z_{xxx} + 72z_x^2z_{xx} = 0\ , \tag{24}$$

which is of second order in time and which for this reason we call bidirectional Satsuma-Hirota (bi-SH) equation. Its stationary reduction is identical to the autonomous case of the F-VI nonlinear ODE, integrated [6] with genus-two hyperelliptic functions.

Therefore, since there exists a path from the (not yet integrated in its full generality) 1:6:8 HH4 case and the (integrated) autonomous F-VI ODE, the general solution of the 1:6:8 can in principle be obtained, this will be addressed in future work.

All the links between the system (19) and other c-KdV systems considered by S. Baker and which reduce to the integrable cases 1:6:1 and 1:6:8 of HH4 are displayed in Figure 2.

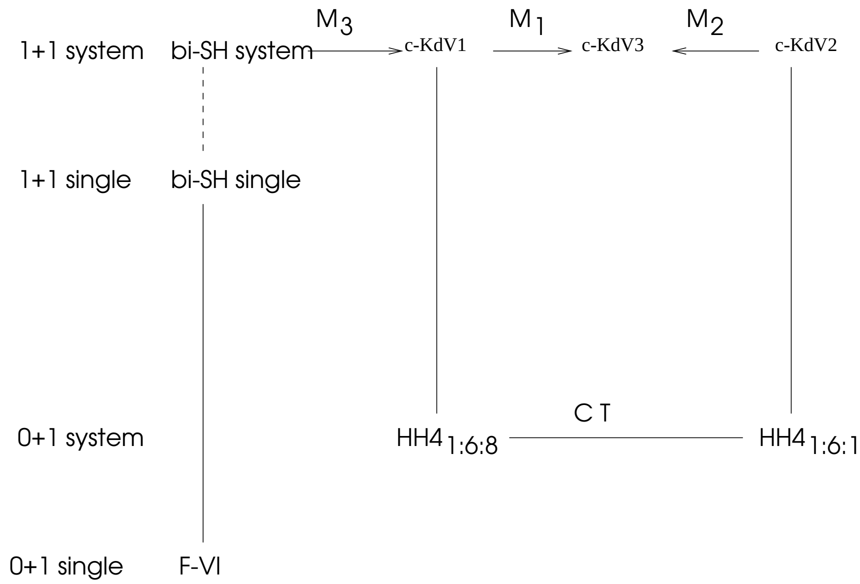

Figure 2. Path from an already integrated ODE (autonomous F-VI) to the quartic cases 1:6:1 and 1:6:8. All 1+1-dimensional systems involved (on the top line) have fourth order Lax pairs. The dashed vertical line from the level "1+1-system" to the level "1+1-single" represents the elimination of one dependent variable. All the other vertical lines represent the stationary reduction. The horizontal lines represent Miura transformations at the level "1+1-system" and canonical transformations at the Hamiltonian level "0+1-system". The systems are defined as (22) for c-KdV$_1$, (13) for c-KdV$_2$, [1, p. 79] for c-KdV$_3$, (19) for the bi-SH system. The Miura maps M$_1$, M$_2$ can be found in [1, Eq. (5.3)] and [1, Eq. (5.8)].

Finally, let us explain the link between the 1:12:16 integrable case of HH4 and two c-KdV systems possessing a fifth order Lax pair, systems respectively equivalent to the bi-SK equation (16) and the bi-KK equation (18).

The following coupled system [9]

$$\left\{ \begin{array}{l} u_t = \left(-2au_{xx} - bu^2 + \dfrac{9a^2}{5b}v\right)_x, \\ v_t = av_{xxx} - bu_{xxxxx} - \dfrac{5b^2}{3a}uu_{xxx} - \dfrac{5b^2}{3a}u_x u_{xx} + buv_x - bu_x v, \end{array} \right. \tag{25}$$

where a, b are nonzero constants, arises from the compatibility condition of the fifth order Lax pair

$$\left\{ \begin{array}{l} \left(\partial_x^5 + \dfrac{5b}{3a}u\partial_x^3 + \dfrac{5b}{3a}u_x\partial_x^2 + v\partial_x\right)\varphi = \lambda\varphi, \\ \left(a\partial_x^3 + bu\partial_x\right)\varphi = \partial_t\varphi. \end{array} \right. \tag{26}$$

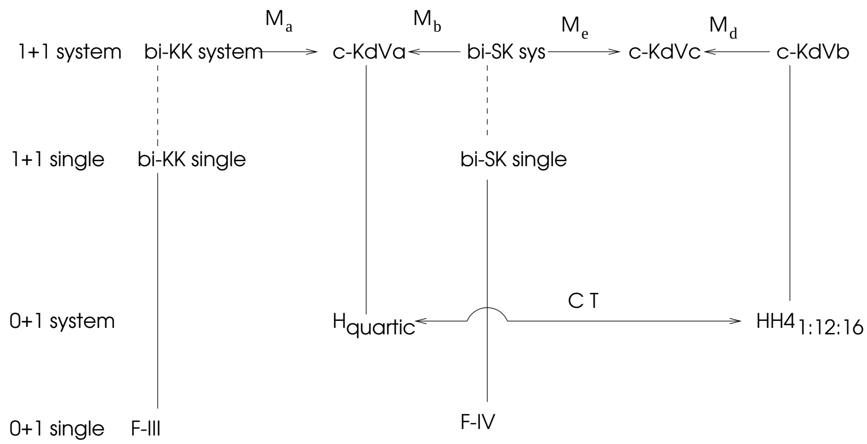

Figure 3. Path from an already integrated ODE (autonomous F-III or F-IV, which are ODEs for q_1 in the HH3-KK and HH3-SK cases) to the quartic case 1:12:16. All 1+1-dimensional systems involved (on the top line) have fifth order Lax pairs. The dashed vertical line from the level "1+1-system" to the level "1+1-single" represents the elimination of one dependent variable. All the other vertical lines represent the stationary reduction. The horizontal lines represent Miura transformations at the level "1+1-system" and birational canonical transformations at the Hamiltonian level "0+1-system". The Miura maps M_a, M_b are given in the text, M_d is given in [1, p. 95], $M_e = M_b M_c$, in which M_c is the Miura transformation from c-KdVa to c-KdVc given in [1, p. 95]. The systems are defined as [1, Eq. (6.9)] for c-KdVb, and [1, p. 95] for c-KdVc.

The field $z = \int u \mathrm{d}x$ of (25) satisfies the sixth order PDE

$$5z_{tt} + \left(5z_{xxt} + 5\frac{b}{a} z_t z_x - a z_{xxxxx} - 5b z_x z_{xxx} - \frac{5b^2}{3a} z_x^3\right)_x = 0, \quad (27)$$

identical to the bi-SK equation (16) for $a = 1,\ b = 3$.

Similarly, the coupled system [9]

$$\begin{cases} u_t = \left(-\dfrac{7}{2} a u_{xx} - b u^2 + \dfrac{9a^2}{5b} v\right)_x, \\ v_t = \dfrac{5}{2} a v_{xxx} - \dfrac{19}{4} b u_{xxxxx} - \dfrac{25b^2}{6a} u u_{xxx} - \dfrac{5b^2}{a} u_x u_{xx} + b u v_x - b u_x v, \end{cases} \quad (28)$$

arises from the compatibility condition of the other fifth order Lax pair

$$\begin{cases} \left(\partial_x^5 + \dfrac{5b}{3a} u \partial_x^3 + \dfrac{5b}{2a} u_x \partial_x^2 + v \partial_x + \dfrac{1}{2} v_x - \dfrac{5b}{12a} u_{xxx}\right) \varphi = \lambda \varphi, \\ \left(a \partial_x^3 + b u \partial_x + \dfrac{b}{2} u_x\right) \varphi = \partial_t \varphi. \end{cases} \quad (29)$$

The field $z = \int u \mathrm{d}x$ satisfies the sixth order PDE

$$5z_{tt} + a\left(5z_{xxt} + 5\frac{b}{a} z_t z_x - a z_{xxxxxx} - 5b z_x z_{xxx} - \frac{5b^2}{3a} z_x^3 - \frac{15b}{4} z_{xx}^2\right)_x = 0, \tag{30}$$

identical to the potential bi-KK equation (18) for $a = 1, b = 3$. The property of these two systems which is of interest to us is the existence of two mappings, respectively (setting $a = 5$), for the system (25) the Miura transformation denoted M_{b}

$$\begin{cases} u_{\mathrm{bi-SK}} = \dfrac{3}{b}(2G + 3F_x - F^2)\ , \\ v_{\mathrm{bi-SK}} = F_{xxx} + G_{xx} - FF_{xx} + GF_x - FG_x + G^2\ , \end{cases} \tag{31}$$

and, for the system (28), the transformation denoted M_{a}

$$\begin{cases} u_{\mathrm{bi-KK}} = \dfrac{3}{b}(2G - 2F_x - F^2), \\ v_{\mathrm{bi-KK}} = -F_{xxx} + 3G_{xx} - FF_{xx} + 2FG_x - F_x^2 + G^2, \end{cases} \tag{32}$$

to a common coupled KdV-type system [1, p. 65] (denoted c-KdV$_a$ in Figure 3)

$$\begin{cases} F_t = \left(-7F_{xx} - 3G_x - 3FF_x - 9FG + 2F^3\right)_x, \\ G_t = 3F_{xxxx} + 2G_{xxx} + 3FG_{xx} - 3F^2 F_{xx} - 3F^2 G_x - 3FF_x^2 \\ \qquad + 3FGF_x + 9F_x F_{xx} + 9F_x G_x + 3GF_{xx} + 3GG_x. \end{cases} \tag{33}$$

This system also possesses a fifth order Lax pair, which can be written in two different ways, either

$$\begin{cases} (\partial_x^2 + F\partial_x + F_x + G)\partial_x(\partial_x^2 - F\partial_x + G)\varphi = \lambda\varphi, \\ \left(5\partial_x^3 + 3(2G - 2F_x - F^2)\partial_x + 3(G_x - F_{xx} - FF_x)\right)\varphi = \partial_t\varphi \end{cases} \tag{34}$$

or

$$\begin{cases} (\partial_x^2 - F\partial_x + G)(\partial_x^2 + F\partial_x + F_x + G)\partial_x\varphi = \lambda\varphi, \\ \left(5\partial_x^3 + 3(2G + 3F_x - F^2)\partial_x\right)\varphi = \partial_t\varphi. \end{cases} \tag{35}$$

It happens that the stationary reduction of (33), which is an unphysical Hamiltonian system [1, pp. 98, 103], is mapped by a canonical transformation to the 1:12:16 case of HH4.

In Figure 3, we display the link between c-KdV systems possessing a fifth order Lax pair and the 1:12:16 integrable Hamiltonian.

5. Conclusion

We have linked each of the three not yet integrated quartic Hénon-Heiles cases to fourth order ODEs recently integrated by Cosgrove, *via* a path involving, on one hand canonical transformations between Hamiltonian systems, and on the other hand Bäcklund transformations between coupled KdV systems. This proves that these three cases have a general solution expressed with hyperelliptic functions of genus two. Their explicit closed form expression will be given in future work.

6. Acknowledgements

The authors acknowledge the financial support of the Tournesol grants T99/040 and T2003.09. MM and RC thank the organizers of this ARW for invitation. CV is a research assistant of the FWO.

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